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Wave solutions of Hopf theories

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Abstract. We present a class of plane-wave solutions for some Hopf theories defined on the symmetric space $SU(2)/U(1)$ in $3+1$ space-time dimensions, using a recently proposed ansatz by Hirayama and Yamashita. These solutions are not solitonic, but they provide us with an example of how plane-wave solutions arise in non-linear field theories.

The Hopf theories are non-linear scalar models in $3+1$ space-time dimensions defined on the symmetric space $SU(2)/U(1)=S^2$. The condition necessary for the Hopf theories to exhibit the static solitons stabilized by the non-zero topological number is that they must be constructed in such a way to circumvent Derrick's scaling theorem [1]. The topological solitons presented in the Hopf theories are classified by the linking number, which characterizes the linking property of the string-like configurations. The solitonic structure of the theories is generally line-like. This differs from the case of instantons or skyrmions, where the soliton number is classified by the winding number and the solitonic structure is point-like.

Generally speaking, all Hopf theories consist of a real three-component vector field $\vec{n}(x)=(n^1,n^2,n^3)$, with unit length $\vec{n}\cdot\vec{n}=1$. If we restrict ourselves to the static field configurations, a topological characterization results. To see this, we observe that for a static field configuration to have a finite energy, the vector field $\vec{n}(\mathbf{r})$ must approach a constant value at spatial infinity, i.e. $\vec{n}(\mathbf{r})\to\vec{n}_0$ as $|\mathbf{r}|\to\infty$. This boundary condition suggests that the Euclidean $\mathbb{R}^3$ space be regarded as a compactified three-dimensional sphere, an S^3 space. Hence, at any fixed time the vector field $\vec{n}(\mathbf{r})$ defines a map that is known as the Hopf map from the 3-sphere space to the 2-sphere target. The solitonic solutions constructed in this way are therefore called hopfions. Note that the spheres between the mapping do not have the same dimensions (S^3 to S^2). Mathematically, such an unusual map falls into the non-trivial homotopy classes, $\pi_3(S^2)=\mathbb{Z}$.

Since the vector field $\vec{n}(x)$ is three-dimensional, we can regard it as the expansion coefficient of an $SU(2)$ Lie-algebra valued vector field $\vec{n}(x)$, that is, $\vec{n}(x)\equiv\sum_{i=1}^{3}n^i(x)\,T_i$. Here, the T_i for $i=1$ to 3 are the generators of the $SU(2)$ Lie algebra. They are defined by $T_i=\frac{1}{2}\sigma_i$ in terms of the Pauli matrices and satisfy $\mathrm{Tr}(T_iT_j)=\frac{1}{2}\delta_{ij}$. Without loss of generality, the vector field $\vec{n}(x)$ can then be identified as the conjugation of the Cartan subalgebra T_3 by a generic group element $U(x)$ in $SU(2)$ [2,3]:

$$\vec{n}(x)\equiv U\,T_3\,U^\dagger\,. \tag{1}$$

It is apparent that in (1) the field $\vec{n}(x)$ remains invariant under a residual $U(1)_{\mathrm{R}}$ gauge transform of the $SU(2)$. This is a right diagonal transformation: $U(x)\to U(x)h(x)$, where $h(x)=\mathrm{e}^{\mathrm{i}\,\alpha(x)\,T_3}$. Therefore, the gauge-invariant fields of $U(x)$ take values in the coset space $SU(2)/U(1)=S^2$ as expected.

Using (1), the derivative of the Lie-algebra valued field $\vec{n}(x)$ with respect to the space-time variable x_μ takes the form

$$\partial_\mu\vec{n}=\mathrm{i}\,[\,\vec{n},\mathbf{R}_\mu\,]\,. \tag{2}$$

In this equation, we introduce the right-invariant Maurer–Cartan covariant vector $\mathbf{R}_\mu$, which is an $SU(2)$ Lie-algebra valued current:

$$\mathbf{R}_\mu\equiv R^i_\mu\,T_i=-\frac{1}{\mathrm{i}}\,\partial_\mu U\,U^\dagger\,, \tag{3}$$

for $i=1,2,3$. By construction, the covariant vector $\mathbf{R}_\mu(x)$ defined in (3) acts as a pure gauge connection, since it satisfies the zero curvature condition, namely, the Maurer–Cartan identity

$$\partial_\mu\mathbf{R}_\nu-\partial_\nu\mathbf{R}_\mu=-\mathrm{i}\,[\mathbf{R}_\mu,\mathbf{R}_\nu]\,. \tag{4}$$

With the definition of the Maurer–Cartan covariant vector $\mathbf{R}_\mu$ in (3), the topological charge, or equivalently the Hopf charge, associated with each field configuration can easily be constructed. The expression for the conserved Hopf charge Q_{H} [3,4] takes the simple form $Q_{\mathrm{H}}=(32\pi^2)^{-1}\epsilon_{ijk}\int\mathrm{d}^3x\,R^3_i(\partial_jR^3_k-\partial_kR^3_j)$, where ϵ_{ijk} is the Levi-Civita tensor. The residual $U(1)_{\mathrm{R}}$ gauge symmetry is evident in this expression.

In this letter, we shall present plane-wave solutions for some Hopf theories defined on the symmetric space

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$SU(2)/U(1) = S^2$ in $3+1$ space-time dimensions, using a recently proposed ansatz by Hirayama and Yamashita [5]. Among the Hopf theories, we are particularly interested in the Nicole model [6], the Aratyn–Ferreira–Zimerman model [7], and the Faddeev–Skyrme model [8]. The details of these models will be presented later. Though the solutions we offer carry no topological number, they provide us with an example on how plane-wave solutions arise in Hopf theories. Originally, the Hirayama–Yamashita ansatz [5] was proposed to construct a class of exact solutions for the $SU(2)$ Skyrme model [9] and the Faddeev–Skyrme model [8]. The covariant field $\mathbf{R}_\mu(x)$ (3) can be given explicitly; yet the final expression for the scalar field $U(x)$ is symbolical since it contains an ordering operation. It is found that these solutions are not solitonic, but with wave characteristics in the Minkowski space [5]. A further extension of the ansatz is reported in [10]. The generalization of the results to the $SU(3)$ Skyrme model can be found in [11].

In essence, we are seeking the solution of the scalar field $U(x)$ given in (1) with the following form [5]:

$$U(x) = U(\xi, \eta), \tag{5}$$

where we define $\xi = k \cdot x$ and $\eta = l \cdot x$. The two constant four-vectors k_μ and l_μ are light-like and arbitrary; that is, they satisfy $k^2 = l^2 = 0$ and $k \cdot l \neq 0$. Using the proposed solution (5), the right current $\mathbf{R}_\mu(x)$ defined in (3) is rewritten as

$$\mathbf{R}_\mu(x) = k_\mu \, \mathbf{A}(\xi, \eta) + l_\mu \, \mathbf{\Lambda}(\xi, \eta). \tag{6}$$

Here, $\mathbf{A}(\xi, \eta)$ and $\mathbf{\Lambda}(\xi, \eta)$ are two Lie-algebra valued fields, which are defined respectively by

$$\mathbf{A}(\xi, \eta) = A_i(\xi, \eta) \, T_i \equiv -\frac{1}{\mathrm{i}} \, \partial_\xi U(\xi, \eta) \, U^\dagger(\xi, \eta), \tag{7}$$

$$\mathbf{\Lambda}(\xi, \eta) = \Lambda_i(\xi, \eta) \, T_i \equiv -\frac{1}{\mathrm{i}} \, \partial_\eta U(\xi, \eta) \, U^\dagger(\xi, \eta). \tag{8}$$

The shorthand notation $\partial_\xi = \frac{\partial}{\partial \xi}$ and $\partial_\eta = \frac{\partial}{\partial \eta}$ is used. We then make a further simplification by assuming that the field $\mathbf{\Lambda}$ in (8) is a constant Lie-algebra valued element [5]. As a result, the Maurer–Cartan identity (2) and the derivative equation of the field $\vec{n}$ (4) are reduced to the following equations:

$$\partial_\eta \mathbf{A}(\xi, \eta) = -\mathrm{i} \, [\, \mathbf{\Lambda}, \mathbf{A}(\xi, \eta) \,] \,, \tag{9}$$

$$\partial_\eta \vec{n}(\xi, \eta) = -\mathrm{i} \, [\, \mathbf{\Lambda}, \vec{n}(\xi, \eta) \,] \,, \tag{10}$$

$$\partial_\xi \vec{n}(\xi, \eta) = -\mathrm{i} \, [\, \mathbf{A}(\xi, \eta), \vec{n}(\xi, \eta) \,] \,. \tag{11}$$

Note that the Lie-algebra valued fields $\mathbf{A}(\xi, \eta)$ and $\vec{n}(\xi, \eta)$, having the same dependence on the η-variable, are not really independent. Both fields are related to each other through the constraint equation (11).

Next, we solve the Lie-algebra valued fields $\mathbf{A}(\xi, \eta)$ and $\vec{n}(\xi, \eta)$ satisfying (9) and (10), respectively, provided that the $\mathbf{\Lambda}$ is a Lie-algebra valued constant. Let us first consider the simplified Maurer–Cartan equation (9). In the same manner, the solution of $\vec{n}(\xi, \eta)$ in (10) can be found. Since (9) is linear in the Lie-algebra valued field $\mathbf{A}(\xi, \eta)$, we write the solution in the form

$$\mathbf{A}(\xi, \eta) = \mathrm{e}^{-\frac{\mathrm{i}}{2}\omega\,\eta} \mathbf{A}(\xi). \tag{12}$$

Then the substitution of this solution (12) into the differential equation (9) results in a set of linear homogeneous equations for the hermitian matrix $(\Lambda_k \mathcal{E}_k)$ as follows:

$$\frac{\omega}{2} \mathbf{A}(\xi) = (\Lambda_k \mathcal{E}_k)_{ij} \, A_j(\xi) \, T_i \,, \tag{13}$$

where the three-dimensional matrices $(\mathcal{E}_k)_{ij} \equiv -\mathrm{i} \, \epsilon_{kij}$ represent the adjoint representation of the $SU(2)$ Lie algebra.

It is easy to determine the eigenvalues and the corresponding eigenvectors of (13). The three eigenvalues are

$$\omega = 0, \pm 2\sqrt{B_2} \,. \tag{14}$$

Here $B_2 = \Lambda_i \Lambda_i$ is the quadratic Casimir invariant out of the Lie-algebra valued constant $\mathbf{\Lambda}$. The three eigenvectors of the eigenequation (13) are constructed as follows. The normalized eigenvector with vanishing eigenvalue $\omega = 0$ is given by

$$\lambda = \frac{1}{\sqrt{B_2}} \mathbf{\Lambda}. \tag{15}$$

The normalized eigenvector of the eigenvalue $\omega = +2\sqrt{B_2}$, designated by $\mathbf{v}$, is

$$\mathbf{v} = \frac{1}{\sqrt{2 B_2 (B_2 - \Lambda_3^2)}} \Big(\Lambda_3 \, \mathbf{\Lambda} - \sqrt{B_2} \, \mathbf{\Lambda_f} - B_2 \, \mathbf{\Delta} \Big) \,, \tag{16}$$

where $\mathbf{\Lambda_f} = \mathrm{i} \, \epsilon_{3ij} \, \Lambda_j \, T_i$ and $\mathbf{\Delta} = \delta_{3i} \, T_i$ are two extra Lie-algebra valued constants. δ_{ij} is the three-dimensional Kronecker delta. Similarly, the corresponding eigenvector with eigenvalue $\omega = -2\sqrt{B_2}$ is the hermitian conjugate of the eigenvector $\mathbf{v}$ (16) and will be denoted by $\bar{\mathbf{v}}$.

It is straightforward to check the orthogonality conditions among these eigenvectors λ, $\mathbf{v}$, and $\bar{\mathbf{v}}$:

$$\mathrm{Tr}(\lambda \, \lambda) = \mathrm{Tr}(\mathbf{v} \, \bar{\mathbf{v}}) = \frac{1}{2}. \tag{17}$$

All other traces are identically zero, for instance, $\mathrm{Tr}(\lambda \, \mathbf{v}) = 0$, $\mathrm{Tr}(\mathbf{v} \, \mathbf{v}) = 0$, and so on.

To obtain the general solution of the Lie-algebra valued field $\mathbf{A}(\xi, \eta)$ fulfilling both (9) and (11), it is useful to calculate the commutation relations among all of the eigenvectors. The results are

$$\begin{aligned} [\lambda, \mathbf{v}] &= \mathbf{v}, \\ [\lambda, \bar{\mathbf{v}}] &= -\bar{\mathbf{v}}, \\ [\mathbf{v}, \bar{\mathbf{v}}] &= \lambda. \end{aligned} \tag{18}$$

Let us give a brief summary of what we have demonstrated concerning the solutions of the simplified Maurer–Cartan identity (9). Since it is linear in the Lie-algebra valued field $\mathbf{A}(\xi, \eta)$, the equation becomes the eigenvalue equation (13). The eigenvalues are found to be 0 and $\pm 2\sqrt{B_2}$, see (14), while the eigenvectors are λ, $\mathbf{v}$ and $\bar{\mathbf{v}}$;

these are presented in (15) and (16). As a result, the general solution for the Lie-algebra valued field $\mathbf{A}(\xi,\eta)$ satisfying (9) can be expressed in the form

$$\begin{aligned}&\mathbf{A}(\xi,\eta) \\ &= f_\lambda(\xi)\,\lambda + f(\xi)\left[\mathrm{e}^{\mathrm{i}\left(\theta(\xi)-\frac{1}{2}\omega\,\eta\right)}\,\mathbf{v} + \mathrm{e}^{-\mathrm{i}\left(\theta(\xi)-\frac{1}{2}\omega\,\eta\right)}\,\bar{\mathbf{v}}\right].\end{aligned} \tag{19}$$

In this general expression, the functions $f_\lambda(\xi)$, $f(\xi)$, and $\theta(\xi)$ are chosen to be real due to the hermiticity of the field $\mathbf{A}(\xi,\eta)$ (7). However, these functions are not arbitrary. They can be determined by using the constraint equation (11) as well as the Lagrange equation of motion of the Hopf theory.

Let us proceed to discuss the general solution of the Lie-algebra valued field $\vec{n}(\xi,\eta)$ in (10). Because it has a form exactly the same as the one of (9) for the field $\mathbf{A}(\xi,\eta)$, we conclude that the general solution of $\vec{n}(\xi,\eta)$ can be written

$$\begin{aligned}&\vec{n}(\xi,\eta) \\ &= g_\lambda(\xi)\,\lambda + g(\xi)\left[\mathrm{e}^{\mathrm{i}\left(\phi(\xi)-\frac{1}{2}\omega\,\eta\right)}\,\mathbf{v} + \mathrm{e}^{-\mathrm{i}\left(\phi(\xi)-\frac{1}{2}\omega\,\eta\right)}\,\bar{\mathbf{v}}\right].\end{aligned} \tag{20}$$

Note that the field $\vec{n}(\xi,\eta)$ defined in (1) is also hermitian, the $g_\lambda(\xi)$, $g(\xi)$, and $\phi(\xi)$ in (20) are again real functions. Furthermore, from the normalization of the field $\vec{n}(\xi,\eta)$ by $\mathrm{Tr}(\vec{n}\,\vec{n}) = \frac{1}{2}$, one infers that

$$g_\lambda(\xi)^2 + 2\,g(\xi)^2 = 1. \tag{21}$$

Thus, we can either choose $|g(\xi)| \leq \sqrt{1/2}$ being a bounded function, or alternatively choose $|g_\lambda(\xi)| \leq 1$ to be bounded.

We have obtained the general solutions for both Lie-algebra valued fields $\mathbf{A}(\xi,\eta)$ and $\vec{n}(\xi,\eta)$ in (19) and (20). As mentioned earlier, these two fields have to satisfy the constraint equation (11). In effect, the constraint produces two relations among the five real functions $f_\lambda(\xi)$, $f(\xi)$, $g(\xi)$, $\theta(\xi)$, and $\phi(\xi)$ introduced above in (19) and (20). For convenience, we take $f(\xi)$ and $\theta(\xi)$ not to be independent in (19). It is found that the modular function $f(\xi)$ of $\mathbf{A}(\xi,\eta)$ can be written in this form:

$$\begin{aligned}&f(\xi)^2 \\ &= \frac{1}{1-2\,g(\xi)^2}\left[\left(g(\xi)'\right)^2 + g(\xi)^2\left(f_\lambda(\xi)+\phi(\xi)'\right)^2\right],\end{aligned} \tag{22}$$

where $g(\xi)' = \partial_\xi g(\xi)$ and $\phi(\xi)' = \partial_\xi\phi(\xi)$. In the same vein, the angular function $\theta(\xi)$ of $\mathbf{A}(\xi,\eta)$ obeys the following expression:

$$\theta(\xi) = \phi(\xi) - \mathrm{arctg}\left[\frac{g(\xi)'}{g(\xi)\left(f_\lambda(\xi)+\phi(\xi)'\right)}\right]. \tag{23}$$

Yet, the functions $f_\lambda(\xi)$, $g(\xi)$, and $\phi(\xi)$ remain arbitrary until a specific form of the Lagrangian is considered.

The discussion on the general solution for the Lie-algebra valued field $\vec{n}(\xi,\eta)$ is useful to construct a class of plane-wave solutions for the Hopf theories. Next, we shall employ this method to study the three particular Hopf theories. For each Hopf theory, by applying the corresponding Lagrange equation of motion, we determine the explicit form for the field $\vec{n}(\xi,\eta)$ in (20), that is, determine the real functions $g(\xi)$ and $\phi(\xi)$. Consequently, the solutions of the field $\mathbf{A}(\xi,\eta)$ can be readily established from the relations (22) and (23). Moreover, the scalar field $U(x)$ given in (1) can be symbolically constructed; that would involve an ordering operation, as shown and discussed in [5].

The first Hopf theory we are interested in is the so-called Nicole model [6]. It is modified from the $O(3)$ non-linear sigma model in an exotic fashion to give a scaling neutral theory. The model is described by the Lagrangian density

$$\mathcal{L}_\mathrm{N} = -\left(-\frac{1}{4}\,\partial_\mu\vec{n}\cdot\partial_\mu\vec{n}\right)^{\frac{3}{2}}, \tag{24}$$

where $\vec{n} = (n^1, n^2, n^3)$ is a three-dimensional vector field with unit length $\vec{n}\cdot\vec{n} = 1$. Note that, without the value $\frac{3}{2}$ of the power in (24), the theory is just the ordinary $O(3)$ non-linear sigma model, which cannot support stable solitons in $3+1$ dimensions due to the usual obstacle of Derrick's scaling theorem [1]. However, the model (24) with the presence of the $\frac{3}{2}$ power will circumvent this obstacle and admit stable topological solitons. The Lagrange equation of motion for the model (24) is derived as

$$\vec{n}\times\left(\vec{n}\times\left[(\partial\,\vec{n}\cdot\partial\,\vec{n})\,\partial^2\vec{n} + \frac{1}{2}\,\partial_\mu\left(\partial\,\vec{n}\cdot\partial\,\vec{n}\right)\partial_\mu\vec{n}\right]\right) = 0\,. \tag{25}$$

If we adopt the ansatz (5), the space-time dependence of the vector field $\vec{n}$ becomes $\vec{n}(x) = \vec{n}(\xi,\eta)$. This choice renders the alternative expression for the equation of motion

$$\begin{aligned}&\vec{n}\times(\vec{n}\times[\,4\,(\partial_\xi\vec{n}\cdot\partial_\eta\vec{n})\,\partial_\xi\partial_\eta\vec{n} \\ &\quad + \partial_\xi\left(\partial_\xi\vec{n}\cdot\partial_\eta\vec{n}\right)\partial_\eta\vec{n} + \partial_\eta\left(\partial_\xi\vec{n}\cdot\partial_\eta\vec{n}\right)\partial_\xi\vec{n}\,]) = 0\,.\end{aligned} \tag{26}$$

Now, according to our construction, the vector field $\vec{n}(\xi,\eta)$ is just the expansion coefficient of the Lie-algebra valued field $\vec{n}(\xi,\eta)$ through the relation $\vec{n} = \sum_{i=1}^{3} n^i T_i$. The plane-wave solutions are then constructed by directly plugging the general solution of $\vec{n}(\xi,\eta)$ (20) into the alternative equation of motion (26). To satisfy this equation of motion, it is found that the two real functions given in (20) have solutions; the function $g(\xi)$ is arbitrary but bounded, $|g(\xi)| \leq \sqrt{1/2}$, while $\phi(\xi) = c$. Here, c is an integral constant. As a result, the Lie-algebra valued field $\vec{n}(\xi,\eta)$ in the Nicole model takes the simple form

$$\begin{aligned}&\vec{n}_\mathrm{N}(\xi,\eta) \\ &= \sqrt{1-2g(\xi)^2}\,\lambda + g(\xi)\left[\mathrm{e}^{\mathrm{i}\left(c-\frac{1}{2}\omega\,\eta\right)}\,\mathbf{v} + \mathrm{e}^{-\mathrm{i}\left(c-\frac{1}{2}\omega\,\eta\right)}\,\bar{\mathbf{v}}\right],\end{aligned} \tag{27}$$

where the subscript "N" denotes the case of the Nicole model. Thus, the Lie-algebra valued field $\mathbf{A}(\xi,\eta)$ (19) can be determined by making use of the relations (22) and (23). The result will be analogous to the expression of the field $\vec{n}_\mathrm{N}(\xi,\eta)$ in (27). Because the function $g(\xi)$ in the formula (27) remains undetermined, the Lie-algebra valued field $\vec{n}_\mathrm{N}(\xi,\eta)$ obtained above represents the class of plane-wave solutions for the Nicole model.

The second Hopf theory to be discussed is the Aratyn–Ferreira–Zimerman model presented in [7], where the model is solved in toroidal coordinates and the solutions are found to represent an infinite number of hopfions. It is described by the Lagrangian density

$$\mathcal{L}_{\mathrm{AFZ}} = -\left([\vec{n}\cdot(\partial_\mu\vec{n}\times\partial_\nu\vec{n})]\,[\vec{n}\cdot(\partial_\mu\vec{n}\times\partial_\nu\vec{n})]\right)^{\frac{3}{4}}\,, \quad (28)$$

where $\vec{n}\cdot\vec{n} = 1$ is assumed. Just like the discussion on the scaling property of the Nicole model (24), the value of the power $\frac{3}{4}$ in the model (28) is introduced such that the theory admits stable solitons. If we define the rank-two antisymmetric tensor

$$H_{\mu\nu} \equiv \vec{n}\cdot(\partial_\mu\vec{n}\times\partial_\nu\vec{n})\,, \quad (29)$$

then Lagrange's equation of motion for the Aratyn–Ferreira–Zimerman model (28) is

$$\vec{n}\times\partial_\mu\left(\partial_\nu\left[(H_{\alpha\beta}H_{\alpha\beta})^{-\frac{1}{4}}\,H_{\mu\nu}\right]\vec{n}\right) = 0\,. \quad (30)$$

The alternative equation of motion can be obtained by imposing the ansatz (5) on the space-time dependence of the vector field $\vec{n}(x) = \vec{n}(\xi,\eta)$. It reads

$$\vec{n}\times(\partial_\eta\vec{n}\,\partial_\xi - \partial_\xi\vec{n}\,\partial_\eta)\,[\vec{n}\cdot(\partial_\xi\vec{n}\times\partial_\eta\vec{n})] = 0\,. \quad (31)$$

Similar to the discussion in the Nicole model, the Lie-algebra valued field $\vec{n}(\xi,\eta)$ has $\vec{n}(\xi,\eta)$ as expansion coefficient. Thus, we obtain the plane-wave solutions of the model by substituting the general form of $\vec{n}(\xi,\eta)$ (20) into the alternative equation of motion (31). This time, the calculation shows that the $g_\lambda(\xi)$ is a linear function in ξ, whereas the $\phi(\xi)$ remains arbitrary. We take $g_\lambda(\xi) = a\,\xi + b$, where a and b are two given constants and $|g_\lambda(\xi)| \le 1$ is bounded. So, the explicit form of the Lie-algebra valued field $\vec{n}(\xi,\eta)$ looks like

$$\begin{aligned}\vec{n}_{\mathrm{AFZ}}(\xi,\eta) &= g_\lambda(\xi)\,\lambda \\ &+ \frac{1}{\sqrt{2}}\left(1 - g_\lambda(\xi)^2\right)^{\frac{1}{2}} \\ &\times\left[\mathrm{e}^{\mathrm{i}\left(\phi(\xi)-\frac{1}{2}\omega\,\eta\right)}\,\mathbf{v} + \mathrm{e}^{-\mathrm{i}\left(\phi(\xi)-\frac{1}{2}\omega\,\eta\right)}\,\bar{\mathbf{v}}\right], \quad (32)\end{aligned}$$

where the subscript "AFZ" denotes the case of the Aratyn–Ferreira–Zimerman model. Furthermore, with the help of the relations (22) and (23), the solution of the Lie-algebra valued field $\mathbf{A}(\xi,\eta)$, (19), can simply be established. Note that the function $\phi(\xi)$ in (32) is arbitrary. Therefore, the expression of the field $\vec{n}_{\mathrm{AFZ}}(\xi,\eta)$ in (32) represents the class of plane-wave solutions for the model (28).

The third and the last model we consider is the Faddeev–Skyrme model [8]. The original motivation for proposing this model is to describe the $SU(2)$ Yang–Mills theory in its low-energy limit [12]. Recent numerical studies have revealed the intricate and fascinating structures of the model [13]. The Lagrangian density of the Faddeev–Skyrme model takes the form

$$\mathcal{L}_{\mathrm{FS}} = m^2\,\partial_\mu\vec{n}\cdot\partial_\mu\vec{n} - \frac{1}{2\,e^2}\,[\vec{n}\cdot(\partial_\mu\vec{n}\times\partial_\nu\vec{n})]^2\,, \quad (33)$$

where m and e are two coupling constants. It is obvious that the first term in (33) represents the ordinary $O(3)$ non-linear sigma model, while the second term, as suggested by the Derrick scaling theorem [1], is introduced to prevent an instability of the field configurations. The equation of motion of the Faddeev–Skyrme model (33) is written as

$$\vec{n}\times\left[\partial_\mu\left(m^2\,(\vec{n}\times\partial_\mu\vec{n}) - \frac{1}{e^2}\,[\vec{n}\cdot(\partial_\mu\vec{n}\times\partial_\nu\vec{n})]\,\partial_\nu\vec{n}\right)\right] = 0. \quad (34)$$

Alternatively, the use of the ansatz (5) on the field $\vec{n}(x) = \vec{n}(\xi,\eta)$ results in the following equation of motion [5]:

$$\begin{aligned}&\vec{n}\times[\partial_\xi\left(\sigma\,(\vec{n}\times\partial_\eta\vec{n}) + [\vec{n}\cdot(\partial_\xi\vec{n}\times\partial_\eta\vec{n})]\,\partial_\eta\vec{n}\right) \quad (35)\\ &\quad + \partial_\eta\left(\sigma\,(\vec{n}\times\partial_\xi\vec{n}) - [\vec{n}\cdot(\partial_\xi\vec{n}\times\partial_\eta\vec{n})]\,\partial_\xi\vec{n}\right)] = 0\,,\end{aligned}$$

where $\sigma = \frac{m^2 e^2}{(l\cdot k)}$ is a combined dimensionless parameter. The plane-wave solutions are obtained by substituting the general solution of $\vec{n}(\xi,\eta)$ (20) into the above equation of motion (35). The results are found to be as follows. The function $g(\xi)$ is a bounded constant $g(\xi) = c_1$, where $|c_1| \le \sqrt{1/2}$ is a constant. In addition, the solution for $\phi(\xi)$ is $\phi(\xi) = c_2$, where c_2 is another constant. Hence, the solution of the Lie-algebra valued fields $\vec{n}(\xi,\eta)$ for the Faddeev–Skyrme model is very simple:

$$\begin{aligned}&\vec{n}_{\mathrm{FS}}(\xi,\eta) \quad (36)\\ &\quad = \sqrt{1 - 2\,c_1^2}\,\lambda + c_1\left[\mathrm{e}^{\mathrm{i}\left(c_2-\frac{1}{2}\omega\,\eta\right)}\,\mathbf{v} + \mathrm{e}^{-\mathrm{i}\left(c_2-\frac{1}{2}\omega\,\eta\right)}\,\bar{\mathbf{v}}\right],\end{aligned}$$

where the subscript "FS" means the case of the Faddeev–Skyrme model. Moreover, by the use of the relations (22) and (23), the solution of the field $\mathbf{A}(\xi,\eta)$ (19) can be gotten with ease. Note that there is no ξ-variable dependence in the formula (36) for the field $\vec{n}_{\mathrm{FS}}(\xi,\eta)$. Nevertheless, the formula (36) still represents the plane-wave solution of the Faddeev–Skyrme model. The ξ-independent feature of the solution can be understood in the following way. First, let us concentrate on the expression of $\vec{n}_{\mathrm{N}}(\xi,\eta)$ (27) of the Nicole model. We observe that the Lagrangian with non-linear sigma term, the term like $(\partial_\mu\vec{n}\cdot\partial_\mu\vec{n})$, will set a very stringent constraint on the angular function $\phi(\xi)$ of the Lie-algebra valued field $\vec{n}(\xi,\eta)$. Similarly, from the solution of $\vec{n}_{\mathrm{AFZ}}(\xi,\eta)$ (32) of the Aratyn–Ferreira–Zimerman model, the fourth derivative term like $[\vec{n}\cdot(\partial_\mu\vec{n}\times\partial_\nu\vec{n})]^2$ will yield another stringent constraint on the modular function $g(\xi)$ of the field $\vec{n}(\xi,\eta)$. Since the Faddeev–Skyrme model contains both terms in the Lagrangian (33), the combinative effect of the both terms consequently gives rise to a very simple form of the solution of the field $\vec{n}_{\mathrm{FS}}(\xi,\eta)$ (36). A result on the plane-wave solutions for the model, which is similar to (36) but using a different approach, has been obtained in [5].

In conclusion, we discuss the applicability of the recently proposed ansatz, the Hirayama–Yamashita ansatz, to the Hopf theories that are defined on the symmetric space $SU(2)/U(1)$. Using the method we have presented, the class of plane-wave solutions for three Hopf theories is separately constructed. The theories considered in this letter are the Nicole model, the Aratyn–Ferreira–Zimerman model and the Faddeev–Skyrme model. Though these solutions are not solitonic, they provide us with an example on how plane-wave solutions arise in non-linear field theories. Let us mention that the method established here can be used to study the non-trivial structure of the plane-wave solutions for the generalized Hopf theories, for example, the model defined on the symmetric space $SU(3)/U(1)^2$ [2].

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